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Poincaré Covariance of Relativistic Quantum Position

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A great number of problems of relativistic position in quantum mechanics are because of the use of coordinates that are not inherent objects of spacetime, cause unnecessary complications, and can lead to misconceptions. We apply a coordinate-free approach to rule out such problems. Thus it will be clear, for example, that the Lorentz covariance of position, required usually on the analogy of Lorentz covariance of spacetime coordinates, is not well posed and we show that in a right setting the Newton–Wigner position is Poincaré covariant.

1. INTRODUCTION

The position observable in relativistic quantum mechanics is an old problem without a fully satisfactory solution; good summaries of the question are given by Kalnay (1971) and Bacry (1988). The trouble is that there is no position observable that has all the natural properties we expect on the base of nonrelativistic quantum mechanics and, moreover, satisfies the requirement of covariance. Earlier position was looked for as a family of Lorentz covariant operators, then projection valued measures or positive operator valued measures were investigated in a system of imprimitivity (Ali, 1998; Wightman, 1962). Recent publications deal with a collection of projections or positive operators, which are related to the structure of spacetime in a convenient way (Bush, 1999; Giannitrapani, 1998). While Wightman considered covariance under Euclidean transformations only, Ali's new approach takes Lorentz covariance into account. However, his treatment involves extra structures whose relation to the problem is not completely clear, as is pointed out in the Discussion.

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Most of the difficulties concerning Lorentz covariance are because of the fact that in all of these treatments spacetime is considered in coordinates. "Much conceptualization in contemporary physics is bogged down by unnecessary assumption concerning a specific choice of coordinates..." (Post, 1971), which results in needless complications and can lead to conceptual errors, too. For instance, it is false to require Lorentz covariance of the position observable on the analogy of Lorentz covariance of spacetime coordinates (see Section 3).

In this paper we put the problem of position observable into a structure of *Spacetime without reference frames*, which eliminates the irrelevant matters, throws new light on the old results, and admits new ones, too.

To illustrate the misleading feature of coordinates, let us recall some usual statements regarding position observable.

1. "The laws of physics should be invariant under transformations of reference frames. This symmetry is guaranteed by postulating the existence of ten infinitesimal generators" of a unitary representation of the Poincaré group (Jordan and Mukunda, 1963).

   It is the free particles that are classified by representations of the Poincaré group; only closed systems have Poincaré symmetry. The equivalence of reference frames is independent of what is described, a closed system or a nonclosed one. If we use *Spacetime without reference frames*, then passive-Poincaré transformations of reference frames will be of no importance, while active-Poincaré transformations are the automorphisms of spacetime and become symmetries of a free system. The confusion of active- and passive-Poincaré transformations yields that one tries to impose the same transformation rule on position coordinates as on the spacetime coordinates.
2. "... it would be difficult to conciliate the operator character of position with the parameter character of time" (Bacry, 1988).

   The use of coordinates confuses some notions: there is spacetime, there are (different) times and (different) spaces according to (different) inertial observers; but position observable (with respect to an observer), whatever it is, though being related to, is not equal to the space of the observer in question. We can define spacetime position as a family of observables with respect to an arbitrary observer $\mathbf{u}$; these observables have a timelike component and a spacelike component relative to an observer $\mathbf{u}'$. The timelike component is a c-number if and only if $\mathbf{u} = \mathbf{u}'$ (see Section 3).
3. The main objection to the Newton–Wigner position (Newton and Wigner, 1949)—besides that it is not Lorentz covariant—is that "localization should also be Lorentz invariant," but it turns out that "if a state is localized for one observer, it is no longer localized for another one," which contradicts Lorentz invariance (Kalnay, 1971).

Lorentz invariance does not mean that something must be the same for all observers. Let us consider a classical mass point: it can be at rest with respect to an observer but this does not imply that it must be at rest with respect to all observers. Replacing "at rest" with "localized," we see that the statement "if a state is localized for one observer, it is no longer localized for another one" does not break Lorentz invariance (see Section 4).

2. SPECIAL RELATIVISTIC SPACETIME MODEL

We shall use *Spacetime without reference frames* introduced by Matolcsi (1993) to investigate the problems of position operator. In such a framework, working with *absolute objects,* i.e. with ones free of coordinates and distinguished observers, we rule out questions regarding Lorentz covariance in the conventional treatments. Although the advantages of this model are well known (Kadianakis, 1991, 1996; Matolcsi, 1985, 1998; Matolcsi and Gruber, 1996), a brief recapitulation of its fundamental concepts is worthwhile.

In usual treatment, spacetime is considered to be $\mathbb{R} \times \mathbb{R}^3$. While spacetime indeed *can* be represented by $\mathbb{R} \times \mathbb{R}^3$, it is also possible to work with less particular mathematical objects. The physical meaning behind $\mathbb{R} \times \mathbb{R}^3$ is fixing an observer, an origin, and some coordinate axes. Thus, in the usual treatment what really happens is the following: one defines the space *and* the time of an observer and then gives transformation rules to change observers. Spacetime as an affine space endowed with some further structure (e.g. Lorentz form) can be well treated mathematically without appealing to $\mathbb{R} \times \mathbb{R}^3$. Instead of giving transformation rules, we can define the notion of an observer and then *calculate* how things seem for different observers.

Let us now formalize the essence of this spacetime model and fix some notations. Let $\mathbf{M}$ be a four-dimensional oriented real vector space, while M is an affine space over $\mathbf{M}$, representing the set of spacetime vectors and spacetime points, respectively. Let $\mathbf{I}$ be a one-dimensional oriented real vector space: the measure line of spacetime distances (thus, for example the time unit S (second) is an element of $\mathbf{I}$). Although spacetime distances could be measured in real numbers after fixing a unit, this would keep us away from talking about the physical dimension of quantities in question.

Further let $\cdot : \mathbf{M} \times \mathbf{M} \to \mathbf{I} \otimes \mathbf{I}$ be symmetrical, bilinear map of the type of 3 plus 1 minus (Lorentz product), endowed with an *arrow orientation* which determines the *future directed* timelike and lightlike vectors. Note that the Lorentz product of two spacetime vectors is an element of $\mathbf{I} \otimes \mathbf{I}$, that is, it has the physical dimension of s^2.

Many times division by time intervals occurs, e.g. in derivation of velocity. Such a procedure is handled properly through the use of the tensorial quotients of vector spaces. Thus an absolute velocity, which is a spacetime vector over a time interval, is an element of $\mathbf{M}/\mathbf{I}$. The Lorentz product can be naturally transferred onto $\mathbf{M}/\mathbf{I}$ where it will be real valued.

The set of absolute velocities is

$$V(1) := \left\{ \mathbf{u} \in \frac{\mathbf{M}}{\mathbf{I}} \,\middle|\, \mathbf{u} \cdot \mathbf{u} = -1, \mathbf{u} \text{ is future directed} \right\}.$$

Given a $\mathbf{u} \in V(1)$, we define

$$\mathbf{E_u} := \{\mathbf{x} \in \mathbf{M} \mid \mathbf{u} \cdot \mathbf{x} = 0\}$$

which is a three-dimensional spacelike linear subspace of $\mathbf{M}$. The restriction of the Lorentz product onto $\mathbf{E_u}$ is an $\mathbf{I} \otimes \mathbf{I}$ valued Euclidean product.

Every spacetime vector can be uniquely split into the sum of a timelike vector parallel to $\mathbf{u}$ and a spacelike vector in $\mathbf{E_u}$, in other words, we can give the $\mathbf{u}$-*splitting*

$$\mathbf{M} \to \mathbf{I} \times \mathbf{E_u}, \qquad \mathbf{x} \mapsto (\boldsymbol{\tau}_\mathbf{u}(\mathbf{x}), \boldsymbol{\pi}_\mathbf{u}(\mathbf{x}))$$

where

$$\boldsymbol{\tau}_\mathbf{u}(\mathbf{x}) := -\mathbf{u} \cdot \mathbf{x}, \qquad \boldsymbol{\pi}_\mathbf{u}(\mathbf{x}) := \mathbf{x} - \boldsymbol{\tau}_\mathbf{u}(\mathbf{x})\mathbf{u}.$$

The best way to formalize our picture about an observer is to define it to be a collection of world lines that satisfies some requirements (e.g. no self-intersections). A point of the space of an observer is in fact a world line. An inertial observer is an observer with only straight, parallel world lines; thus an inertial observer can be given by an absolute velocity $\mathbf{u} \in V(1)$. According to Einstein's synchronization, spacetime points x and y are $\mathbf{u}$-*simultaneous* if and only if $\mathbf{u} \cdot (x - y) = 0$, in other words, $x - y \in \mathbf{E_u}$. Thus $\mathbf{u}$-simultaneous spacetime points form an affine hyperplane over $\mathbf{E_u}$. A $\mathbf{u}$-simultaneous hyperplane is considered to be a $\mathbf{u}$-*instant* and the set $I_\mathbf{u}$ of such hyperplanes is the time of the observer, briefly the $\mathbf{u}$-*time*. The time interval between $\mathbf{u}$-instants t_1 and t_2 is defined to be

$$t_1 - t_2 := \boldsymbol{\tau}_\mathbf{u}(x_1 - x_2) \quad (x_1 \in t_1, x_2 \in t_2)$$

which is a good definition as it is independent of the choice of x_1 and x_2. $I_\mathbf{u}$ endowed with this subtraction is an affine space over $\mathbf{I}$.

The space points of the inertial observer $\mathbf{u}$ are straight lines in spacetime, parallel to $\mathbf{u}$. The space of the observer $\mathbf{u}$, denoted by $E_\mathbf{u}$, endowed with the subtraction

$$q_1 - q_2 := \boldsymbol{\pi}_\mathbf{u}(x_1 - x_2) \quad (x_1 \in q_1, x_2 \in q_2)$$

is an affine space over the vector space $\mathbf{E_u}$ (the definition is independent of the choice of x_1 and x_2).

The Lorentz group is

$$\mathcal{L} := \{\mathbf{L} : \mathbf{M} \to \mathbf{M} \mid \mathbf{L} \text{ is linear}, \mathbf{Lx} \cdot \mathbf{Ly} = \mathbf{x} \cdot \mathbf{y}\,(\mathbf{x}, \mathbf{y} \in \mathbf{M})\}.$$

Orthochronous Lorentz transformations preserve the arrow orientation of the Lorentz form.

The three-dimensional orthogonal group is not a subgroup of the Lorentz group (contrarily to the usual statement in the coordinatized treatment). For all $\mathbf{u} \in V(1)$,

$$\mathcal{O}_{\mathbf{u}} := \{\mathbf{L} \in \mathcal{L} \mid \mathbf{L}_{\mathbf{u}} = \mathbf{u}\}$$

is a subgroup of the Lorentz group, which is isomorphic to the three-dimensional orthogonal group (in fact the restrictions of the elements of $\mathcal{O}_{\mathbf{u}}$ onto the three dimensional Euclidean space $\mathbf{E}_{\mathbf{u}}$ are orthogonal maps). $\mathcal{O}_{\mathbf{u}}$ and $\mathcal{O}_{\mathbf{u}'}$ are different if $\mathbf{u} \neq \mathbf{u}'$.

Similarly, the time inversion and the space inversion are not elements of the Lorentz group. For all $\mathbf{u} \in V(1)$ we can give the **u**-time inversion and the **u**-space inversion:

$$\mathbf{x} \mapsto -\boldsymbol{\tau}_{\mathbf{u}}(\mathbf{x})\mathbf{u} + \pi_{\mathbf{u}}(\mathbf{x}), \qquad \mathbf{x} \mapsto \boldsymbol{\tau}_{\mathbf{u}}(\mathbf{x})\mathbf{u} - \pi_{\mathbf{u}}(\mathbf{x}).$$

The Poincaré group is

$$\mathcal{P} := \{L : M \to M \mid L \text{ is affine}, \ \mathbf{L} \in \mathcal{L}\}$$

where $\mathbf{L}$ denotes the linear map under L. A Poincaré transformation over an orthochronous Lorentz transformation is called orthochronous.

The Lorentz group is not a subgroup of the Poincaré group (contrarily to the usual statement in the coordinatized treatment); it cannot be, because Lorentz transformations are $\mathbf{M} \to \mathbf{M}$ linear maps, Poincaré transformations are $M \to M$ affine maps. For all $o \in M$,

$$\mathcal{L}_o := \{L \in \mathcal{P} \mid L(o) = o\}$$

is a subgroup isomorphic to the Lorentz group, but $\mathcal{L}_o$ and $\mathcal{L}_{o'}$ are different for different o and o'. The elements of $\mathcal{L}_o$ are called o-homogeneous Poincaré transformations.

Of course, neither the time inversion nor the space inversion are elements of the Poincaré group. We can only define a time inversion with respect to an observer **u** and a time (a **u**-instant t).

For all $\mathbf{u} \in V(1)$ and $t \in I_{\mathbf{u}}$

$$\mathcal{E}_{\mathbf{u},t} := \{L \in \mathcal{P} \mid L[t] = t\}$$

is a subgroup of the Poincaré group; the restriction of its elements onto t are Euclidean transformations of the hyperplane t; moreover, it contains the **u**-time inversion with respect to the **u**-instant t.

3. POSITION OBSERVABLE(S)

We investigate the position observable of a free particle; thus we accept that an irreducible unitary ray representation U of the spacetime symmetry group (Poincaré group) is given on the Hilbert space of states of the particle.

We mention that in the absolute description a state is in fact a *process* (a history of what happens). The representation reflects the fact that a spacetime symmetry as a transformation turns a possible process of a closed system into another possible process (thus, the representation does not refer to changes of coordinate systems).

A convenient way to describe physical quantities such as position is to use projection valued measures or positive operator valued measures. Wightman (1962) defined localization, i.e. position of a free particle as a projection valued measure P defined on the Borel subsets of space such that $U_S P(E) U_S^{-1} = P(S[E])$ for all Borel subsets E of space and for S being an arbitrary Euclidean transformation in space or the time inversion, where U is the corresponding representation of the Poincaré group.

Because neither the space nor the Euclidean subgroup of the Poincaré group nor the time inversion exist, we reformulate this approach in our framework as follows:

Consider an observer $\mathbf{u}$ and a $\mathbf{u}$-instant t. For every Borel set $E \in \mathcal{B}(t)$ there should be a projection $P_{\mathbf{u},t}(E)$ standing for the event of the particle being located in E. By the natural expectations of localization, $P_{\mathbf{u},t}$ is required to be a projection valued measure having the following connection with the representation of the Poincaré group.

$$U_S P_{\mathbf{u},t}(E) U_S^{-1} = P_{\mathbf{u},t}(S[E]) \tag{1}$$

for all $E \in \mathcal{B}(t)$ and $S \in \mathcal{E}_{\mathbf{u},t}$. Because we only want to deal with a one particle system, in the following we will always consider an irreducible representation of the Poincaré group.

Applying Wightman's proof, we can state that for fixed $\mathbf{u}$ and t, a projection valued measure satisfying (1) is unique under some regularity conditions.

Note that we have *many* spacelike hypersurfaces, and of course, localization on one of them is not the same as on another one. Furthermore, the transformation rule (1) says nothing about the relation between $P_{\mathbf{u},t}$ and $P_{\mathbf{u}',t'}$ for $\mathbf{u}' \neq \mathbf{u}$ or $t' \neq t$. Nevertheless, the following nice transformation property can be shown:

Proposition 1. *Let an imprimitivity system (1) be given for all* $\mathbf{u} \in V(1)$ *and* $t \in I_{\mathbf{u}}$. *If Wightman's regularity condition holds then*

$$U_L P_{\mathbf{u},t}(E) U_L^{-1} = P_{L\mathbf{u},L[t]}(L[E]) \tag{2}$$

for all $\mathbf{u} \in V(1)$, $t \in I_{\mathbf{u}}$, *Borel subset* E *of* t *and for all orthochronous Poincaré transformations* L.

Proof: Let L be fixed; putting

$$\bar{P}_{\mathbf{u},t}(E) := U_L^{-1} P_{\mathbf{Lu},L[t]}(L[E])U_L$$

we find that $\bar{P}_{\mathbf{u},t}$ satisfies (1), because if $S \in \mathcal{E}_{\mathbf{u},t}$ then LSL^{-1} is in $\mathcal{E}_{\mathbf{Lu},L[t]}$. The regularity condition for $\bar{P}_{\mathbf{u},t}$ trivially holds. As a consequence of uniqueness, we have the desired result. □

It is known that integrating the space coordinates by Wightman's projection valued measure, one gets the Newton–Wigner position.

Accordingly, by choosing a spacetime origin o, with the aid of the above projection valued measure we can construct a family of position operators:

$$W^o_{\mathbf{u},t} := \int_t (\mathrm{id}_t - o)\, \mathrm{d}P_{u,t} \quad (o \in M, \mathbf{u} \in V(1), t \in I_{\mathbf{u}}).$$

$W^o_{\mathbf{u},t}$ is an $\mathbf{M}$ valued totally self-adjoint vector operator, which we call the o-centered generalized Newton–Wigner position at the $\mathbf{u}$-instant t.

Using the transformation properties of integration by projection valued measure we can easily find the transformation rule of the members of the family of generalized Newton–Wigner positions:

Proposition 2.

$$U_L W^o_{\mathbf{u},t} U_L^{-1} = \mathbf{L}^{-1} W^{Lo}_{\mathbf{Lu},L[t]} \tag{3}$$

We now understand that the above equality is the Poincaré covariance of the generalized Newton–Wigner position. We emphasize that this Poincaré covariance of the family of positions does not refer to the equivalence of reference frames; it reflects the properties of the particle according to what has been said in the beginning of the current Section.

It is important to see that $W^o_{\mathbf{u},t}$ is a "four-vector" ($\mathbf{M}$-valued) but it does not transform as a spacetime-vector, i.e. for a fixed $\mathbf{u}$, t and spacetime origin $o \in t$ (which corresponds to the usual considerations in coordinates), $Q := W^o_{\mathbf{u},t}$ is not a "four-vector operator": $U_L^{-1} Q U_L \neq \mathbf{L} Q$ for an o-homogeneous Poincaré transformation L.

The $\mathbf{u}$-spacelike component of $W^o_{\mathbf{u},t}$ corresponds to the original Newton–Wigner position. It is interesting, however, that we can consider its $\mathbf{u}'$-spacelike components, too. Applying (3), we easily find the following:

Proposition 3. *The $\mathbf{u}'$-spacelike component of $W^o_{\mathbf{u},t}$ transforms as a $\mathbf{u}'$-spacevector, that is,*

$$U_L \boldsymbol{\pi}_{\mathbf{u}'}\left(W^o_{\mathbf{u},t}\right) U_L^{-1} = \mathbf{R}^{-1} \boldsymbol{\pi}_{\mathbf{u}'}\left(W^o_{\mathbf{u},t}\right)$$

for o-homogeneous $L \in \mathcal{E}_{\mathbf{u}',t'}$ *if and only if* $\mathbf{u} = \mathbf{u}'$, *where* $\mathbf{R}$ *is the restriction of* $\mathbf{L}$ *onto* $E_{\mathbf{u}'}$ *(a rotation in* $\mathbf{E}_{\mathbf{u}'}$*).*

The generalized Newton–Wigner position has timelike component, too, for which we derive the following interesting result.

Proposition 4. *The* $\mathbf{u}'$*-timelike component of* $W^{o}_{\mathbf{u},t}$ *is a c-number if and only if* $\mathbf{u} = \mathbf{u}'$.

Proof: Using the properties of integration by projection valued measures, it is easy to see that the $\mathbf{u}'$-timelike component is a c-number if and only if $\boldsymbol{\tau}_{\mathbf{u}}(\mathrm{id}_t - o)$ is constant almost everywhere according to $P_{\mathbf{u},t}$. It is constant only on the two-dimensional affine subspaces of t parallel to $\mathbf{E}_{\mathbf{u}} \cap \mathbf{E}_{\mathbf{u}'}$. But considering the transformation rules (2), it is impossible that the support of $P_{\mathbf{u},t}$ is in one of these subspaces. □

4. LOCALIZATION AND CAUSALITY

Let us investigate localization problem in our framework. We conceive that a state Φ (i.e. an element of the Hilbert space) is *localized* in a set $E \in \mathcal{B}(t)$ at a $\mathbf{u}$-instant t if $P_{\mathbf{u},t}(E)\Phi = \Phi$ holds. Poincaré invariance of localization means that if Φ is localized in E at a $\mathbf{u}$-instant t and L is a proper Poincaré transformation then $U_L\Phi$ is localized at the $\mathbf{Lu}$-instant $L[t]$ in $L[E]$, which trivially holds.

Now it is clear that the requirement of Lorentz invariance, “if a state is localized for one observer, it must be localized for all other ones” is not well posed. Lorentz invariance—or rather Poincaré invariance—should mean that *if a state is localized for one observer then a Poincaré transform of the state must be localized for the correspondingly transformed observer*.

By causality, we expect that if Φ is localized in $E \in \mathcal{B}(t)$ then Φ is localized in $(E + T) \cap t' \in \mathcal{B}(t')$, that is, $P_{\mathbf{u}',t'}((E + T) \cap t')\Phi = \Phi$ holds for every observer $\mathbf{u}'$ and $\mathbf{u}'$-instant t', where T denotes the cone of timelike vectors. *The existence of a state localized for one observer and not localized for another one*, i.e., the existence of a state Φ such that $P_{\mathbf{u},t}(E)\Phi = \Phi$ for a $t \in I_{\mathbf{u}}$ but $P_{\mathbf{u}',t'}((E + T) \cap t')\Phi \neq \Phi$ for some $\mathbf{u}'$-instant t' *denies causality but not the Poincaré invariance*.

The acausal feature of the Dirac equation is well known and thoroughly treated in the literature (Hegerfeldt, 1985; Hegerfeldt and Ruijsenaars, 1980; Ruijsenaars, 1981).

Causality requirement yields that $P_{\mathbf{u},t}(E)$ and $P_{\mathbf{u}',t'}(E')$ should be orthogonal if E and E' are spacelike separated. It is known that a projection valued map defined on the bounded Borel subsets of spacelike hyperplanes satisfying covariant transformation rules and causality ($[P_{\mathbf{u},t}(E), P_{\mathbf{u}',t'}(E')] = 0$) is equal to zero (Bush, 1999; Malament, 1996). That is why the generalized Newton–Wigner position violates causality, though being Poincaré covariant.

5. DISCUSSION

In this paper we have investigated an old problem in relativistic quantum mechanics: to find Poincaré covariant position operator. We have used a special relativistic spacetime model free of distinguished observers and reference frames. With the aid of this formalism it is obvious how physical quantities such as position are connected to observers of spacetime.

For different observers, position corresponds to localization on different, not even parallel hypersurfaces; and for a single observer but different time instants, it corresponds to localization on parallel but still not equal hypersurfaces.

Therefore, instead of a *single* position, we have a Poincaré covariant family of position operators, the generalized Newton–Wigner position, labelled by observers, time instants, and spacetime origins. Each member of the family is an **M** valued vector operator whose spacelike and timelike components behave differently for different observers.

The power of the spacetime model is reflected in the fact that Wightman's uniqueness statement on a system of imprimitivity is the only tool in deriving our result; further, we emphasize that our treatment involves only projection valued measures (and avoids positive operator valued measures).

A nice Lorentz covariance theorem of position was given by Ali (1998). In that paper, however, position is based on positive operator valued measures and it needs the notion of generalized systems of covariance. As a tool in the proof a tight family of coherent states is used, which is extraneous to the problem of position. Moreover, because a coordinatized formulation is used, the role of the observers is not clarified, only one hyperplane (corresponding to the "zero time point" coordinate) is considered for each observer, and spacetime origins do not appear—that is why spacetime translations are not treated.

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